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Normal geodesics in stationary Lorentzian manifolds with unbounded coefficients

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Abstract

Let $\mathcal{M}$ be a stationary manifold equipped with a Lorentz metric whose coefficients are unbounded. By using variational methods and topological tools, some existence and multiplicity results of normal geodesics joining two fixed submanifolds can be proved.

Keywords: Stationary Lorentzian manifolds; Normal geodesics; Saddle point theorem; Relative category

1. Introduction and main results

Let $\mathcal{M}$ be a smooth finite-dimensional manifold and $\langle\cdot,\cdot\rangle_z$ be a Lorentz metric on it, i.e., a smooth symmetric (0, 2) tensor field on $\mathcal{M}$ which defines a non-degenerate bilinear form of index 1 on each tangent space $T_z\mathcal{M}$, $z \in \mathcal{M}$. A smooth curve $z : [0, 1] \to \mathcal{M}$ is a geodesic in $\mathcal{M}$ if

$$D_s\dot{z}(s) = 0 \qquad \text{for all } s \in [0, 1],$$

where D_s denotes the covariant derivative along z induced by the Levi-Civita connection of the metric $\langle\cdot,\cdot\rangle_z$. It is well known that, if $z = z(s)$ is a geodesic, then its energy $E(z) = \langle\dot{z}(s), \dot{z}(s)\rangle_z$ is constant in [0, 1]. So, a geodesic is named timelike, lightlike or spacelike if its energy is negative, null or positive, respectively (for more details, see [1,16,20]).

Here, we are interested in geodesics joining two submanifolds in a special class of Lorentzian manifolds, so the following definitions hold.

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Definition 1.1. Let $(\mathcal{M}, \langle\cdot,\cdot\rangle_z)$, $\mathcal{N}_0$, $\mathcal{N}_1$ be a Lorentzian manifold and, respectively, two of its submanifolds. A curve $z : [0, 1] \to \mathcal{M}$ is a normal geodesic joining $\mathcal{N}_0$ to $\mathcal{N}_1$ if it is a geodesic such that

$$z(0) \in \mathcal{N}_0, \quad \dot{z}(0) \in T_{z(0)}\mathcal{N}_0^{\perp} \qquad \text{and} \qquad z(1) \in \mathcal{N}_1, \quad \dot{z}(1) \in T_{z(1)}\mathcal{N}_1^{\perp}, \tag{1.1}$$

where for $i = 0, 1$, $T_{z(i)}\mathcal{N}_i^{\perp}$ denotes the orthogonal space of $T_{z(i)}\mathcal{N}_i$ in $T_{z(i)}\mathcal{M}$ with respect to the non-degenerate bilinear form $\langle\cdot,\cdot\rangle_z$.

Definition 1.2. A Lorentzian manifold $(\mathcal{M}, \langle\cdot,\cdot\rangle_z)$ is called (standard) stationary if there exists a smooth connected finite-dimensional Riemannian manifold $(\mathcal{M}_0, \langle\cdot,\cdot\rangle_x)$ such that $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ and $\langle\cdot,\cdot\rangle_z$ is given by

$$\langle\zeta, \zeta\rangle_z = \langle\alpha(x)\xi, \xi\rangle_x + 2\langle\delta(x), \xi\rangle_x\tau - \beta(x)\tau^2 \tag{1.2}$$

for any $z = (x, t) \in \mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ and $\zeta = (\xi, \tau) \in T_z\mathcal{M} \equiv T_x\mathcal{M}_0 \times \mathbb{R}$, where $\alpha(x)$ is a smooth symmetric linear strictly positive operator from $T_x\mathcal{M}_0$ into itself, δ a smooth vector field and β a smooth and positive scalar field on the Riemannian manifold $\mathcal{M}_0$.

In particular, if $\delta \equiv 0$, the metric (1.2) is called static and $(\mathcal{M}, \langle\cdot,\cdot\rangle_z)$ is a static Lorentzian manifold.

From now on, let $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ be a stationary Lorentzian manifold equipped with the metric (1.2). Let P_0 and P_1 be two given submanifolds of $\mathcal{M}_0$ and let $t_0, t^* \in \mathbb{R}$ be fixed. Set

$$\tilde{P}_0 = P_0 \times \{t_0\}, \qquad \tilde{P}_1 = P_1 \times \mathbb{R}, \qquad \tilde{P}_1^* = P_1 \times \{t^*\}. \tag{1.3}$$

The aim of this paper is to study the existence of geodesics $z : [0, 1] \to \mathcal{M}$ joining in a normal way $\tilde{P}_0$ to $\tilde{P}_1$ or, respectively, $\tilde{P}_0$ to $\tilde{P}_1^*$.

In particular, it means to study the existence of geodesics joining a point to a worldline of an observer or the geodesic connectedness in a stationary manifold.

If the coefficients α, β and δ of the stationary metric are bounded, some existence results for lightlike geodesics joining an event to a line have been studied in [10], while the existence of lightlike geodesics joining $\tilde{P}_0$ to $\tilde{P}_1$ (normal only "in the spatial part") has been stated in [6]; furthermore, other existence results for normal geodesics joining particular submanifolds have been stated in static manifolds (cf. [19]) or in orthogonal splitting type ones, i.e., when in (1.2) it is $\delta \equiv 0$ while α, β are time dependent (cf. [5]).

On the other hand, we know only two results concerning the geodesical connectedness of a stationary manifold with unbounded coefficients (cf. [14,23]). In both these papers it is $\alpha \equiv 1$ while δ has a sublinear growth at infinity with respect to the Riemannian metric on $\mathcal{M}_0$, but the difference is in the assumptions on β and in the methods: in [23] β has a sublinear growth at infinity and the author proves the existence of a geodesic joining two fixed points by using a linking argument applied to the action functional, while in [14] β has to be bounded from above but the stationary manifold so obtained is just an example of a more general class of Lorentzian manifolds which an intrinsic approach applies to.

Here, we want to extend the existence result in [23] to geodesics joining two given submanifolds. Moreover, at least in the case of the geodesics from $\tilde{P}_0$ to $\tilde{P}_1$, we weaken the assumptions on β just requiring a subquadratic growth. Then, some multiplicity theorems are proved.

Let us point out that if the coefficients of $\langle\cdot,\cdot\rangle_z$ are bounded, then the problem can be reduced to the research of critical points of a new functional bounded from below and depending only on the Riemannian part (for example, see [6,13]). On the contrary, in this paper the lack of upper bounds for the coefficients does not make useful such a "trick". So, it is better to manage directly the action functional and use a finite-dimensional approximation on the space of the time variable in order to apply a generalization of the Rabinowitz saddle point theorem and the theory of relative category.

Let us remark that the hypothesis 'β subquadratic' is not too strange since, if β has a quadratic growth, a geodesical connectedness result may not hold. In fact, a counterexample is given by the anti-de Sitter space–time $\mathcal{M} =]-\pi/2, \pi/2[\times\mathbb{R}$ equipped with the metric

$$\mathrm{d}s^2 = \frac{1}{\cos^2 x}\,\mathrm{d}x^2 - \frac{1}{\cos^2 x}\,\mathrm{d}t^2,$$

which is geodesically complete, but not geodesically connected (cf. [22]).

We will state the following results:

Theorem 1.3. *Let $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ be a manifold equipped with the stationary Lorentzian metric (1.2) such that*

(H_1) *$(\mathcal{M}_0, \langle\cdot,\cdot\rangle_x)$ is a complete C^3 n-dimensional Riemannian manifold;*
(H_2) *there exist $q \in [0, 1[$, some strictly positive constants λ, ν, R_1, R_2 and a point $x_0 \in \mathcal{M}_0$ such that for all $x \in \mathcal{M}_0$, $\xi \in T_x\mathcal{M}_0$, it is*

$$\langle\alpha(x)\xi, \xi\rangle_x \geq \lambda\langle\xi, \xi\rangle_x, \qquad \beta(x) \geq \nu,$$

$$\beta(x) \leq R_1 + R_2 d^{q+1}(x, x_0), \tag{1.4}$$

$$\sqrt{\langle\delta(x), \delta(x)\rangle_x} \leq R_1 + R_2 d^q(x, x_0), \tag{1.5}$$

where $d(\cdot,\cdot)$ denotes the distance in $\mathcal{M}_0$ induced by its Riemannian metric.

Let P_0 and P_1 be two subsets of $\mathcal{M}_0$ satisfying the following conditions:

(H_3) *P_0 and P_1 are closed submanifolds of $\mathcal{M}_0$ such that one of them is compact;*
(H_4) *$P_0 \cap P_1 = \emptyset$.*

Then, there exists at least a normal geodesic joining $\tilde{P}_0$ to $\tilde{P}_1$.

Theorem 1.4. *Let $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ be a manifold equipped with the stationary Lorentz metric (1.2) such that the hypothesis (H_1) holds while assumption (H_2) is replaced by the stronger condition:*

$(H_2)^*$ *there exist $q \in [0, 1[$, some strictly positive constants λ, ν, R_1, R_2 and a point $x_0 \in \mathcal{M}_0$ such that for all $x \in \mathcal{M}_0$, $\xi \in T_x\mathcal{M}_0$, the conditions (1.4) and (1.5) are satisfied and*

$$\beta(x) \leq R_1 + R_2 d^q(x, x_0). \tag{1.6}$$

Let P_0 and P_1 be two subsets of $\mathcal{M}_0$ satisfying (H_3) and (H_4).

Then, there exists a normal geodesic joining $\tilde{P}_0$ *to* $\tilde{P}_1^*$. *Moreover, if* $|t^*|$ *is small enough such a geodesic is spacelike while it is timelike if* $|t^*|$ *is large enough in the stronger assumption* $q \in [0, 1/2[$.

In particular, if we assume $P_0 = \{x_1\}$ and $P_1 = \{x_2\}$ ($x_1, x_2 \in \mathcal{M}_0$), the assumption (H_3) is trivial while (H_4) means $x_1 \neq x_2$. Then, the previous theorem implies the main result in [23] (if it is $\alpha(x) \equiv 1$) or, in general, the following corollary:

Corollary 1.5. *If* $(\mathcal{M}, \langle\cdot,\cdot\rangle_z)$ *is a stationary Lorentzian manifold such that* (H_1) *and* $(H_2)^*$ *hold, then* $\mathcal{M}$ *is geodesically connected.*

Remark 1.6. By the assumptions (H_1) and (H_2), respectively $(H_2)^*$, it follows that $\mathcal{M}$ is globally hyperbolic (cf. [25, Corollary 3.4]). Anyway, this is not enough to imply that $\mathcal{M}$ is geodesically connected (for a counterexample, see [26]).

The following multiplicity theorems hold even if, eventually, we consider two submanifolds P_0 and P_1 which are not disjoint:

Theorem 1.7. *Let* $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ *be a manifold equipped with the stationary Lorentz metric* (1.2) *which satisfies* (H_1), (H_2) *and*

(H_5) *there exists* $p \geq 0$ *such that for all* $x \in \mathcal{M}_0$, $\xi \in T_x\mathcal{M}_0$, *it is*

$$\langle\alpha(x)\xi,\xi\rangle_x \leq (R_1 + R_2 d^p(x, x_0))\langle\xi,\xi\rangle_x,$$

where $R_1, R_2 > 0$ *and* x_0 *are as in the hypothesis* (H_2).

Let P_0 *and* P_1 *be two subsets of* $\mathcal{M}_0$ *such that* (H_3) *holds and assume*

(H_6) $\mathcal{M}_0$ *is not contractible in itself while both* P_0 *and* P_1 *are contractible in the whole manifold* $\mathcal{M}_0$.

Then, there exist infinitely many non-constant spacelike geodesics z_n *joining* $\tilde{P}_0$ *to* $\tilde{P}_1$ *whose energies* $E(z_n)$ *are such that*

$$\lim_{n\to+\infty} E(z_n) = +\infty. \tag{1.7}$$

Theorem 1.8. *Let* $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ *be a manifold equipped with the stationary Lorentz metric* (1.2) *such that* (H_1), $(H_2)^*$ *and* (H_5) *are satisfied. Let* P_0 *and* P_1 *be two subsets of* $\mathcal{M}_0$ *which satisfy* (H_3). *If* (H_6) *holds too, then there exist infinitely many spacelike geodesics* z_n *joining* $\tilde{P}_0$ *to* $\tilde{P}_1^*$ *which verify* (1.7).

On the other hand,

$$\lim_{|t^*|\to+\infty} N(P_0, P_1, t^*) = +\infty, \tag{1.8}$$

where $N(P_0, P_1, t^*)$ *denotes the number of timelike orthogonal geodesics joining* $\tilde{P}_0$ to $\tilde{P}_1^*$.

Remark 1.9. In the hypotheses (H_2), $(H_2)^*$ and (H_5) it is not restrictive to assume that R_1 and R_2 are the same constant. Moreover, since the real number p in the assumption (H_5) can be arbitrarily large, we suppose $p > 2q$.

Remark 1.10. Let us point out that, if $P_0 \cap P_1 \neq \emptyset$, then for every $\bar{x} \in P_0 \cap P_1$ the constant function $\bar{z} = (\bar{x}, 0)$ is a trivial normal geodesic joining $\tilde{P}_0$ to $\tilde{P}_1$. Moreover, if $t^* = 0$, such a trivial geodesic $\bar{z}$ also joins $\tilde{P}_0$ and $\tilde{P}_1^*$. So, the assumption (H_4) in Theorem 1.3 or in Theorem 1.4 with $t^* = 0$ implies that the found geodesic is not trivial, while if $t^* \neq 0$ and small enough, (H_4) allows to state that the found geodesic is spacelike. On the other hand, this assumption is not necessary in the multiplicity theorems because, obviously, the condition (1.7) gives the existence of infinitely many non-constant spacelike normal geodesics.

Remark 1.11. The previous results apply, in particular, if P_0 and P_1 are reduced to a single point. Then, Theorems 1.3 and 1.7 imply an existence and, respectively, a multiplicity result for geodesics joining one point to a "line", while Theorems 1.4 and 1.8 give an existence and, respectively, a multiplicity result for geodesics joining two fixed points.

2. Variational approach

First of all, we need some functional manifolds in order to look for normal geodesics joining two submanifolds via variational methods.

Let $I = [0, 1]$ and $n \in \mathbb{N}$. The Sobolev space $H^1(I, \mathbb{R}^n)$ is the set of the absolutely continuous curves with square summable derivative equipped with the scalar product

$$(x, y) = \int_0^1 \langle \dot{x}, \dot{y} \rangle \, \mathrm{d}s + \int_0^1 \langle x, y \rangle \, \mathrm{d}s$$

and the norm

$$\|x\|_{1,2}^2 = \|x\|^2 + \|\dot{x}\|^2 = \int_a^b |x(s)|^2 \, \mathrm{d}s + \int_a^b |\dot{x}(s)|^2 \, \mathrm{d}s,$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product of $\mathbb{R}^n$ and $\| \cdot \|$ the usual norm of $L^2(I, \mathbb{R}^n)$.

Let $\mathcal{M}$ be a connected, finite-dimensional smooth manifold. We denote by $H^1(I, \mathcal{M})$ the set of curves $z : I \to \mathcal{M}$ such that for any local chart (U, φ) of $\mathcal{M}$, with $U \cap z(I) \neq \emptyset$, the curve $\varphi \circ z$ belongs to the Sobolev space $H^1(z^{-1}(U), \mathbb{R}^n)$, $n = \dim \mathcal{M}$.

It is well known (cf. [21]) that $H^1(I, \mathcal{M})$ is equipped with a structure of infinite-dimensional manifold modeled on the Hilbert space $H^1(I, \mathbb{R}^n)$. If $z \in H^1(I, \mathcal{M})$, the tangent space to $H^1(I, \mathcal{M})$ at z can be identified as follows:

$$T_z H^1(I, \mathcal{M}) \equiv \{\zeta \in H^1(I, T\mathcal{M}) : \pi \circ \zeta = z\},$$

where $T\mathcal{M}$ denotes the tangent bundle of $\mathcal{M}$ and $\pi : T\mathcal{M} \to \mathcal{M}$ the bundle projection. In other words, $T_z H^1(I, \mathcal{M})$ is the set of the vector fields along z whose components with respect to a local chart are functions of class H^1.

If $\mathcal{M}$ is a Lorentzian manifold equipped with the metric $\langle \cdot, \cdot \rangle_z$, the action integral $f : H^1(I, \mathcal{M}) \to \mathbb{R}$ can be defined as

$$f(z) = \int_0^1 \langle \dot{z}(s), \dot{z}(s) \rangle_z \, \mathrm{d}s, \quad z \in H^1(I, \mathcal{M}). \tag{2.1}$$

It is easy to prove that f is a C^1 functional and for any $z \in H^1(I, \mathcal{M})$ and $\zeta \in T_z H^1(I, \mathcal{M})$ there results

$$\mathrm{d}f(z)[\zeta] = 2\int_0^1 \langle \dot{z}(s), D_s\zeta(s)\rangle_z \,\mathrm{d}s. \tag{2.2}$$

Let $\mathcal{N}_0$ and $\mathcal{N}_1$ be two submanifolds of $\mathcal{M}$ and set

$$\Omega(\mathcal{N}_0, \mathcal{N}_1; \mathcal{M}) = \{z \in H^1(I, \mathcal{M}) : z(0) \in \mathcal{N}_0,\ \ z(1) \in \mathcal{N}_1\}. \tag{2.3}$$

Since $\Omega(\mathcal{N}_0, \mathcal{N}_1; \mathcal{M})$ is a smooth submanifold of $H^1(I, \mathcal{M})$ (cf. [17]), taken $z \in \Omega(\mathcal{N}_0, \mathcal{N}_1; \mathcal{M})$ the tangent space at z to $\Omega(\mathcal{N}_0, \mathcal{N}_1; \mathcal{M})$ is given by

$$T_z\Omega(\mathcal{N}_0, \mathcal{N}_1; \mathcal{M}) = \{\zeta \in T_z H^1(I, \mathcal{M}) : \zeta(0) \in T_{z(0)}\mathcal{N}_0,\ \ \zeta(1) \in T_{z(1)}\mathcal{N}_1\}.$$

According to Definition 1.1, the geodesics joining $\mathcal{N}_0$ to $\mathcal{N}_1$ which are normal with respect to $\langle\cdot,\cdot\rangle_z$ satisfy a suitable variational principle. Indeed, if we denote by $\bar{f}$ the restriction of the action integral f to the manifold $\Omega(\mathcal{N}_0, \mathcal{N}_1; \mathcal{M})$, the following proposition shows that, as in the Riemannian case (cf. [17]), the critical points z of $\bar{f}$ on $\Omega(\mathcal{N}_0, \mathcal{N}_1; \mathcal{M})$ are the normal geodesics joining $\mathcal{N}_0$ to $\mathcal{N}_1$.

Proposition 2.1. *A curve $z : I \to \mathcal{M}$ is a normal geodesic joining $\mathcal{N}_0$ to $\mathcal{N}_1$ if and only if $z \in \Omega(\mathcal{N}_0, \mathcal{N}_1; \mathcal{M})$ is a critical point of $\bar{f}$.*

Proof. Taken $z \in \Omega(\mathcal{N}_0, \mathcal{N}_1; \mathcal{M})$, simple calculations give

$$\mathrm{d}\bar{f}(z)[\zeta] = -2\int_0^1 \langle D_s\dot{z}, \zeta\rangle_z \,\mathrm{d}s + 2\langle \dot{z}(1), \zeta(1)\rangle_z - 2\langle \dot{z}(0), \zeta(0)\rangle_z \tag{2.4}$$

for all $\zeta \in T_z\Omega(\mathcal{N}_0, \mathcal{N}_1; \mathcal{M})$. So, if z is a normal geodesic from $\mathcal{N}_0$ to $\mathcal{N}_1$, then (1.1) and the geodesic equation imply that $z \in \Omega(\mathcal{N}_0, \mathcal{N}_1; \mathcal{M})$ and $\mathrm{d}\bar{f}(z) = 0$.

Vice versa, if $z \in \Omega(\mathcal{N}_0, \mathcal{N}_1; \mathcal{M})$ is a critical point of $\bar{f}$, then there results

$$\int_0^1 \langle D_s\dot{z}, \zeta\rangle_z \,\mathrm{d}s = 0 \quad \text{for all } \zeta \in T_z\Omega(\mathcal{N}_0, \mathcal{N}_1; \mathcal{M}) \text{ with compact support}.$$

Hence, by standard density and regularity arguments, z is a geodesic in $\mathcal{M}$. Thus, by (2.4) it follows $\langle \dot{z}(1), \zeta(1)\rangle_z = \langle \dot{z}(0), \zeta(0)\rangle_z$ for all $\zeta \in T_z\Omega(\mathcal{N}_0, \mathcal{N}_1; \mathcal{M})$; whence, choosing in particular ζ such that $\zeta(0) = 0$ or $\zeta(1) = 0$, the boundary conditions (1.1) follow. □

From now on, let $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ be a stationary Lorentz manifold with $(\mathcal{M}_0, \langle\cdot,\cdot\rangle_x)$ Riemannian manifold (according to Definition 1.2). By the Nash embedding theorem we can assume that $\mathcal{M}_0$ is a submanifold of an Euclidean space $\mathbb{R}^N$ while $\langle\cdot,\cdot\rangle_x$ is the restriction to $\mathcal{M}_0$ of the Euclidean metric $\langle\cdot,\cdot\rangle$ of $\mathbb{R}^N$. So, in the sequel, we shall still denote by $\langle\cdot,\cdot\rangle$ the Euclidean metric on $\mathcal{M}_0$ and by d the corresponding distance (see Remark 4.3 or, in general, [18]). By means of the product structure of $\mathcal{M}$, the infinite-dimensional manifold $H^1(I, \mathcal{M})$ is diffeomorphic to the product manifold $H^1(I, \mathcal{M}_0) \times H^1(I, \mathbb{R})$. Moreover, $H^1(I, \mathcal{M})$ is equipped with a structure of an infinite-dimensional Riemannian manifold

$\langle\cdot,\cdot\rangle_1$ by setting

$$\langle\zeta,\zeta\rangle_1=\int_0^1\langle\xi,\xi\rangle\,\mathrm{d}s+\int_0^1\langle D_s\xi,D_s\xi\rangle\,\mathrm{d}s+\int_0^1\tau^2\,\mathrm{d}s+\int_0^1\dot{\tau}^2\,\mathrm{d}s$$

for any $z=(x,t)\in H^1(I,\mathcal{M})$ and $\zeta=(\xi,\tau)\in T_zH^1(I,\mathcal{M})\equiv T_xH^1(I,\mathcal{M}_0)\times T_tH^1(I,\mathbb{R})\equiv T_xH^1(I,\mathcal{M}_0)\times H^1(I,\mathbb{R})$. Since $(\mathcal{M}_0,\langle\cdot,\cdot\rangle)$ is a complete Riemannian manifold, also $H^1(I,\mathcal{M})$ is a complete Riemannian manifold equipped with the previous scalar product (cf. [21]).

Then, by (1.2) the action integral $f:H^1(I,\mathcal{M})\to\mathbb{R}$ in (2.1) becomes

$$f(z)=\int_0^1(\langle\alpha(x)\dot{x},\dot{x}\rangle+2\langle\delta(x),\dot{x}\rangle\dot{t}-\beta(x)\dot{t}^2)\,\mathrm{d}s \tag{2.5}$$

for any $z=(x,t)\in H^1(I,\mathcal{M})$.

Let P_0, P_1 be two closed submanifolds of $\mathcal{M}_0$ and consider the submanifolds $\tilde{P}_1$ and $\tilde{P}_1^*$ of $\mathcal{M}$ defined in (1.3).

Moreover, since if $z=(x,t)$ is a geodesic in $\mathcal{M}$ then $z_T=(x,t+T)$ is still a geodesic for any $T\in\mathbb{R}$, without loss of generality we can assume $t_0=0$ and define $\tilde{P}_0=P_0\times\{0\}$.

Remark 2.2. Let $\mathcal{N}$ be a submanifold of $\mathcal{M}$ while $z=(x,t)$ is a geodesic on $\mathcal{M}$ satisfying the conditions

$$z(s_0)\in\mathcal{N},\qquad \dot{z}(s_0)\in T_{z(s_0)}\mathcal{N}^{\perp} \tag{2.6}$$

for a certain $s_0\in[0,1]$. If P is a submanifold of $\mathcal{M}_0$ and there exists $\bar{t}\in\mathbb{R}$ such that $\mathcal{N}=P\times\{\bar{t}\}$, then the conditions (2.6) are equivalent to

$$x(s_0)\in P,\quad t(s_0)=\bar{t},$$
$$\langle\alpha(x(s_0))\dot{x}(s_0)+\dot{t}(s_0)\delta(x(s_0)),\xi\rangle_x=0\quad\text{for all }\xi\in T_{x(s_0)}P.$$

On the other hand, if $\mathcal{N}=P\times\mathbb{R}$, (2.6) becomes

$$x(s_0)\in P,$$
$$\langle\alpha(x(s_0))\dot{x}(s_0)+\dot{t}(s_0)\delta(x(s_0)),\xi\rangle_x=0\quad\text{for all }\xi\in T_{x(s_0)}P,$$
$$\langle\delta(x(s_0)),\dot{x}(s_0)\rangle_x-\dot{t}(s_0)\beta(x(s_0))=0.$$

In order to look for geodesics joining $\tilde{P}_0$ to $\tilde{P}_1$, set $Z^0=\Omega(\tilde{P}_0,\tilde{P}_1;\mathcal{M})$. According to the product structure of such submanifolds, there results

$$Z^0\equiv\Omega(P_0,P_1;\mathcal{M}_0)\times W^0,$$

where W^0 is the closed subspace of $H^1(I,\mathbb{R})$ defined as

$$W^0=\{t\in H^1(I,\mathbb{R}):t(0)=0\}.$$

Moreover, the tangent space at a curve $z=(x,t)\in Z^0$ is given by

$$T_zZ^0=T_z\Omega(\tilde{P}_0,\tilde{P}_1;\mathcal{M})\equiv T_x\Omega(P_0,P_1;\mathcal{M}_0)\times W^0.$$

It is easy to see that

$$W^0 = H_0^1 \oplus \mathbb{R} j_I$$

with

$$H_0^1 = \{\tau \in H^1(I, \mathbb{R}) : \tau(0) = \tau(1) = 0\}, \qquad j_I : s \in I \mapsto s \in \mathbb{R}.$$

Whence, by the Poincaré inequality the space W^0 can be equipped with the following equivalent norm:

$$\|t\|_0^2 = \|\dot{t}\|^2 = \int_0^1 \dot{t}^2 \, \mathrm{d}s. \tag{2.7}$$

Now, assume $Z^* = \Omega(\tilde{P}_0, \tilde{P}_1^*; \mathcal{M})$. By the product structure of $\tilde{P}_0$ and $\tilde{P}_1^*$ it follows

$$Z^* \equiv \Omega(P_0, P_1; \mathcal{M}) \times W^*,$$

where

$$W^* = \{t \in H^1(I, \mathbb{R}) : t(0) = 0, \ \ t(1) = t^*\}.$$

Clearly, W^* is a closed affine submanifold of $H^1(I, \mathbb{R})$ as

$$W^* = H_0^1 + T^* \quad \text{with} \ \ T^* : s \in I \mapsto t^* s \in \mathbb{R}.$$

Hence, the tangent space at a curve $z = (x, t) \in Z^*$ is given by

$$T_z Z^* = T_z \Omega(\tilde{P}_0, \tilde{P}_1^*; \mathcal{M}) \equiv T_x \Omega(P_0, P_1; \mathcal{M}_0) \times H_0^1.$$

Let us remark that, if the assumption (H_3) holds, both the submanifolds Z^0 and Z^* of $H^1(I, \mathcal{M})$ can be equipped with the following equivalent Riemannian structure

$$\langle \zeta, \zeta \rangle_H = \langle (\xi, \tau), (\xi, \tau) \rangle_H = \int_0^1 \langle D_s \xi, D_s \xi \rangle \, \mathrm{d}s + \int_0^1 \dot{\tau}^2 \, \mathrm{d}s.$$

Finally, we set

$$f^0 = f|_{Z^0}, \qquad f^* = f|_{Z^*}.$$

Then, Proposition 2.1 implies that the normal geodesics joining $\tilde{P}_0$ to $\tilde{P}_1$, respectively $\tilde{P}_1^*$, are the critical points of f^0 on Z^0, respectively f^* on Z^*.

3. Critical point theorems for indefinite functionals

In this section we present some abstract critical point theorems for indefinite functionals. First, we recall the Palais–Smale condition.

Definition 3.1. Let Z be a Riemannian manifold. A C^1 functional $f : Z \to \mathbb{R}$ satisfies the Palais–Smale condition, briefly (PS), if every sequence $(z_n)_{n\in\mathbb{N}}$ in Z such that

$$\sup_{n\in\mathbb{N}} |f(z_n)| < +\infty \quad \text{and} \quad \lim_{n\to+\infty} f'(z_n) = 0$$

has a convergent subsequence (here, $f'(z_n)$ goes to 0 in the norm induced on the cotangent bundle by the Riemannian metric on Z).

An existence result for critical points can be obtained by a slight variant of the classical saddle point theorem (cf. [3,24]).

Theorem 3.2. *Let Ω be a complete Riemannian manifold and H a separable Hilbert space. Fixed a linear subspace H_0 of H and an element $T \in H$, let $(a_n)_{n\in\mathbb{N}}$ be an orthonormal basis of H_0. Set $W = H_0 + T$ and $Z = \Omega \times W$. Let $f : Z \to \mathbb{R}$ be a C^1 functional and, for any integer $k \geq 1$, define*

$$W_k = \text{span}\{a_n : n = 1, 2, \dots, k\} + T, \qquad Z_k = \Omega \times W_k \quad \text{and} \quad f_k = f|_{Z_k}.$$

Fix $\bar{t} \in H_0$ and $\bar{x} \in \Omega$. For any real positive number R consider the sets:

$$S = \{(x, \bar{t} + T) \in Z : x \in \Omega\} = \Omega \times \{\bar{t} + T\},$$
$$Q(R) = \{(\bar{x}, t) \in Z : \|t - T - \bar{t}\| \leq R\},$$

where $\|\cdot\|$ is the norm of the Hilbert space H.

Assume that f_k satisfies the (PS) condition for any $k \in \mathbb{N}$ and there exists $R > 0$ such that

$$\sup f(Q(R)) < +\infty, \qquad \sup f(\partial Q(R)) < \inf f(S).$$

Then, for any $k \in \mathbb{N}$, $k \geq 1$, f_k has a critical level $c_k \in [\inf f(S), \sup f(Q(R))]$, where

$$c_k = \inf_{h\in\Gamma_k} \sup_{x\in Q_k(R)} f_k(h(x)),$$
$$\Gamma_k = \{h \in C(Z_k, Z_k) : h(z) = z \quad \text{for all } z \in \partial Q_k(R)\}$$

and

$$Q_k(R) = \{(\bar{x}, t) \in Z_k : \|t - T - \bar{t}\| \leq R\}.$$

Proof. Let $k \in \mathbb{N}$ be fixed. We remark that $S \subset W_k$; moreover, $Q_k(R) \subset Q(R)$ and $\partial Q_k(R) \subset \partial Q(R)$ imply that

$$\sup f_k(Q_k(R)) \leq \sup f(Q(R)), \qquad \sup f_k(\partial Q_k(R)) \leq \sup f(\partial Q(R)).$$

According to the saddle point theorem (see, e.g., [18, Theorem 8.3.1]) it follows that c_k is a critical level of f_k such that

$$\inf f(S) \leq c_k \leq \sup f(Q_k(R)) \leq \sup f(Q(R)). \qquad \square$$

Now, in order to state a multiplicity result, we need the notion of relative category and its main properties (cf. [7,11,12,27]).

Definition 3.3. Let Y and A be closed subsets of a topological space Z. The category of A in Z relative to Y, briefly $\mathrm{cat}_{Z,Y}(A)$, is the least integer n such that there exist $n+1$ closed subsets of Z, $A_0, A_1, \dots, A_n$, $A = A_0 \cup A_1 \cup \dots \cup A_n$, and $n+1$ functions, $h_j \in C([0,1] \times A_j, Z)$, $j = 0, 1, \dots, n$, such that

(i) $h_j(0, z) = z$ for $z \in A_j$, $0 \le j \le n$;
(ii) $h_0(1, z) \in Y$ for $z \in A_0$, and $h_0(\sigma, y) \in Y$ for all $y \in A_0 \cap Y$, $\sigma \in [0, 1]$;
(iii) $h_j(1, z) = z_j$ for $z \in A_j$ and some $z_j \in Z$, $1 \le j \le n$.

If a finite number of such sets does not exist, we set $\mathrm{cat}_{Z,Y}(A) = +\infty$.

Clearly, $\mathrm{cat}_Z(A) = \mathrm{cat}_{Z,\emptyset}(A)$ is the classical Ljusternik–Schnirelman category of A in Z.

Proposition 3.4. *Let A, B, Y be closed subsets of a topological space Z.*

(i) *If $A \subset B$ then* $\mathrm{cat}_{Z,Y}(A) \le \mathrm{cat}_{Z,Y}(B)$;
(ii) $\mathrm{cat}_{Z,Y}(A \cup B) \le \mathrm{cat}_{Z,Y}(A) + \mathrm{cat}_Z(B)$;
(iii) *if there exists $h \in C([0,1] \times A, Z)$ such that $h(\sigma, y) = y$ for $y \in A \cap Y$ and $\sigma \in [0, 1]$, then* $\mathrm{cat}_{Z,Y}(A) \le \mathrm{cat}_{Z,Y}(B)$, *where $B = \overline{h(1, A)}$.*

Remark 3.5. Let Z be a topological space and Y a closed subset of Z. Then (ii) of Proposition 3.4 implies that the relative category and the classical Ljusternik–Schnirelman category are connected by the inequality

$$\mathrm{cat}_{Z,Y}(A) \le \mathrm{cat}_Z(A) \quad \text{for any closed set } A \subset Z.$$

It is easy to see that Definition 3.3 implies the following proposition.

Proposition 3.6. *Let Z be a topological space and C, Λ be two subsets of Z such that C is a closed strong deformation retract of $Z \setminus \Lambda$, i.e., there exists a continuous map $\mathcal{R} : [0,1] \times (Z \setminus \Lambda) \to Z$ such that*

$$\mathcal{R}(0, z) = z \quad \textit{for all } z \in Z \setminus \Lambda,$$
$$\mathcal{R}(1, z) \in C \quad \textit{for all } z \in Z \setminus \Lambda,$$
$$\mathcal{R}(\sigma, z) = z \quad \textit{for all } z \in C, \sigma \in [0, 1].$$

Then, $\mathrm{cat}_{Z,C}(Z \setminus \Lambda) = 0$.

In the sequel, we will need the following additional property of the relative category (for the proof, cf. [4, Proposition 2.2]).

Proposition 3.7. *Let Y, Z', Y' be closed subsets of a topological space Z such that $Y' \subset Z'$. Suppose that there exist a retraction $r : Z \to Z'$, i.e., a continuous map such that $r(z) = z$*

for all $z \in Z'$, *and a homeomorphism* $\Phi : Z \to Z$ *such that*

(i) $\Phi(Y') \subset Y$;
(ii) $r \circ \Phi^{-1}(Y) \subset Y'$.

Then, if A' *is a closed subset of* Z', *it results that*

$$\mathrm{cat}_{Z,Y}(\Phi(A')) \geq \mathrm{cat}_{Z',Y'}(A').$$

The following theorem gives a multiplicity result for critical levels of a strongly indefinite functional (for more details, see [4]).

Theorem 3.8. *Let* Z *be a* C^2 *complete Riemannian manifold modeled on a Hilbert space and let* $f : Z \to \mathbb{R}$ *be a* C^1 *functional which satisfies the* (PS) *condition. Let us assume that there exist two subsets* Λ *and* C *of* Z *such that* C *is a closed strong deformation retract of* $Z \backslash \Lambda$, *and*

$$\inf_{z \in \Lambda} f(z) > \sup_{z \in C} f(z), \qquad \mathrm{cat}_{Z,C}(Z) > 0.$$

Then, f *has at least* $\mathrm{cat}_{Z,C}(Z)$ *critical points in* Z *whose critical levels are greater than or equal to* $\inf f(\Lambda)$. *Moreover, if* $\mathrm{cat}_{Z,C}(Z) = +\infty$, *there exists a sequence* $(z_m)_{m \in \mathbb{N}}$ *of critical points of* f *such that*

$$\lim_{m \to +\infty} f(z_m) = \sup_{z \in Z} f(z).$$

Remark 3.9. In Theorem 3.8 the critical levels c_m are characterized as follows:

$$c_m = \inf_{B \in F_m} \sup_{z \in B} f(z) \quad \text{for any } 1 \leq m \leq \mathrm{cat}_{Z,C}(Z),$$

where

$$F_m = \{B \subset Z : B \text{ closed}, \quad \mathrm{cat}_{Z,C}(B) \geq m\}.$$

Since we want to apply Theorem 3.8 in order to get multiplicity results for normal geodesics joining two submanifolds in a stationary Lorentzian manifold, the following result concerning the topological properties of the space of curves joining the fixed submanifolds is basic.

Theorem 3.10. *Let* $\mathcal{M}_0$ *be a simply connected and non-contractible smooth manifold; let* P_0, P_1 *be two closed submanifolds of* $\mathcal{M}_0$ *and assume that* P_0 *and* P_1 *are contractible in* $\mathcal{M}_0$. *Denote by* $\Omega(P_0, P_1; \mathcal{M}_0)$ *the space of curves of class* H^1 *joining* P_0 *to* P_1 *in* $\mathcal{M}_0$ (*cf. the definition* (2.3)). *Let* D^k, S^k *be the unit disk in* $\mathbb{R}^k$ *and, respectively, its boundary.*

Then, for any $m \in \mathbb{N}$, *there exists a compact set* $\Gamma_{k,m} \subset \Omega(P_0, P_1; \mathcal{M}_0) \times D^k$ *such that*

$$\mathrm{cat}_{\Omega(P_0,P_1;\mathcal{M}_0)\times D^k,\, \Omega(P_0,P_1;\mathcal{M}_0)\times S^k}(\Gamma_{k,m}) \geq m.$$

The proof of this result is a consequence of Corollary 4.6 in [9] and Proposition 3.2 in [8].

Remark 3.11. Let $(\Gamma_{k,m})_{k,m}$ be the family of compact subsets of the manifold $\Omega(P_0, P_1; \mathcal{M}_0) \times D^k$ which exists by Theorem 3.10. Fix $m \in \mathbb{N}$. The arguments used in [8] show that the sets $\Gamma_{k,m}$ have the same projection on $\Omega(P_0, P_1; \mathcal{M}_0)$ for all $k \in \mathbb{N}$.

4. Existence results

From now on, we fix a stationary Lorentz manifold $(\mathcal{M}, \langle\cdot,\cdot\rangle_z)$ satisfying (H_1); moreover, let P_0 and P_1 be such that (H_3) holds and define $\tilde{P}_0$, $\tilde{P}_1$ and $\tilde{P}_1^*$ as in (1.3) with $t_0 = 0$. For simplicity, set $\Omega(P_0, P_1) = \Omega(P_0, P_1; \mathcal{M}_0)$.

As we have seen in Section 2, normal geodesics joining $\tilde{P}_0$ to $\tilde{P}_1$, respectively $\tilde{P}_1^*$, are critical points of f^0 on Z^0, respectively f^* on Z^*.

Remark 4.1. Since the action functional f in (2.5) is Fréchet differentiable, it is easy to prove that by (2.2) its Fréchet differential at $z = (x, t) \in X = \Omega(P_0, P_1) \times H^1(I, \mathbb{R})$ in $\zeta = (\xi, \tau) \in T_z X \equiv T_x\Omega(P_0, P_1) \times H^1(I, \mathbb{R})$ is given by

$$\begin{aligned} \mathrm{d}f(z)[\zeta] = {} & 2\int_0^1 \langle\alpha(x)\dot{x}, \dot{\xi}\rangle\,\mathrm{d}s + \int_0^1 \langle\alpha'(x)[\xi]\dot{x}, \dot{x}\rangle\,\mathrm{d}s + 2\int_0^1 \langle\delta'(x)[\xi], \dot{x}\rangle\dot{t}\,\mathrm{d}s \\ & + 2\int_0^1 \langle\delta(x), \dot{\xi}\rangle\dot{t}\,\mathrm{d}s + 2\int_0^1 \langle\delta(x), \dot{x}\rangle\dot{\tau}\,\mathrm{d}s - \int_0^1 \beta'(x)[\xi]\dot{t}^2\,\mathrm{d}s \\ & - 2\int_0^1 \beta(x)\dot{t}\dot{\tau}\,\mathrm{d}s, \end{aligned}$$

where α', β' and δ' denote, respectively, the derivatives of α, β and δ with respect to the Riemannian structure on $\mathcal{M}_0$.

In order to apply the abstract theorems of the previous section, we need the following results.

Proposition 4.2. *Assume that (1.4) holds. Then the functional f^0 satisfies the (PS) condition on Z^0.*

Proof. Let $(z_n)_{n\in\mathbb{N}}$ be a (PS) sequence in Z^0, i.e.,

$$\sup_{n\in\mathbb{N}} |f^0(z_n)| < +\infty, \tag{4.1}$$

$$\lim_{n\to+\infty} \mathrm{d}f^0(z_n) = 0, \tag{4.2}$$

where $\mathrm{d}f^0(z_n)$ goes to 0 in the norm induced on the cotangent bundle by the Riemannian metric on Z^0.

Set $\tau_n = t_n \in W^0 = T_{t_n}W^0$. By (4.2) there exists a real sequence $\varepsilon_n \to 0$ as $n \to +\infty$ such that

$$\begin{aligned} \varepsilon_n\|\dot{t}_n\| = \mathrm{d}f^0(z_n)[(0, t_n)] &= 2\int_0^1 (\langle\delta(x_n), \dot{x}_n\rangle\dot{t}_n - \beta(x_n)\dot{t}_n^2)\,\mathrm{d}s \\ &= f^0(z_n) - \int_0^1 (\langle\alpha(x_n)\dot{x}_n, \dot{x}_n\rangle + \beta(x_n)\dot{t}_n^2)\,\mathrm{d}s. \end{aligned}$$

By the previous equalities and (4.1) it follows that there exists a real constant M such that

$$\int_0^1 (\langle\alpha(x_n)\dot{x}_n, \dot{x}_n\rangle + \beta(x_n)\dot{t}_n^2)\,\mathrm{d}s \le M - \varepsilon_n\|\dot{t}_n\|,$$

which implies, by (1.4), that

$$\left(\int_0^1 \langle\dot{x}_n, \dot{x}_n\rangle\,\mathrm{d}s\right)_{n\in\mathbb{N}} \quad \text{and} \quad \left(\int_0^1 \dot{t}_n^2\,\mathrm{d}s\right)_{n\in\mathbb{N}} \quad \text{are bounded.} \tag{4.3}$$

Since P_0 or P_1 is compact, (4.3) means that the sequence $(z_n)_{n\in\mathbb{N}}$ is bounded in Z^0, there exists a curve $z = (x, t)$ such that, up to a subsequence,

$$z_n \rightharpoonup z \ \text{ weakly in } H^1(I, \mathbb{R}^N) \times H^1(I, \mathbb{R}), \qquad z_n \to z \ \text{ uniformly in } I.$$

Clearly, it is $z \in Z^0$, since both P_0 and P_1 are closed in the complete manifold $\mathcal{M}_0$. By [2, Lemma 2.1], there exist two sequences $(\xi_n)_{n\in\mathbb{N}}$, $(\nu_n)_{n\in\mathbb{N}} \subset H^1(I, \mathbb{R}^N)$, $\xi_n \in T_{x_n}\Omega(P_0, P_1)$, such that $x_n - x = \xi_n + \nu_n$ for all $n \in \mathbb{N}$, while

$$\xi_n \rightharpoonup 0 \ \text{ weakly} \quad \text{and} \quad \nu_n \to 0 \text{ strongly in } \ H^1(I, \mathbb{R}^N). \tag{4.4}$$

Taking $\tau_n = t - t_n \in W^0$, we have

$$\tau_n \rightharpoonup 0 \ \text{ weakly in } H^1(I, \mathbb{R}), \tag{4.5}$$

so (4.2) gives

$$\lim_{n\to+\infty} \mathrm{d}f^0(z_n)[(\xi_n, \tau_n)] = 0. \tag{4.6}$$

Obviously, $(\alpha'(x_n))_{n\in\mathbb{N}}$, $(\beta'(x_n))_{n\in\mathbb{N}}$ and $(\delta'(x_n))_{n\in\mathbb{N}}$ are bounded; moreover, $(\xi_n)_{n\in\mathbb{N}}$ and $(\tau_n)_{n\in\mathbb{N}}$ converge uniformly to 0. Then, according to (4.3) it follows that

$$\int_0^1 \langle\alpha'(x_n)[\xi_n]\dot{x}_n, \dot{x}_n\rangle\,\mathrm{d}s = \mathrm{o}(1), \qquad \int_0^1 \langle\delta'(x_n)[\xi_n], \dot{x}_n\rangle\dot{t}_n\,\mathrm{d}s = \mathrm{o}(1),$$

$$\int_0^1 \beta'(x_n)[\xi_n]\dot{t}_n^2\,\mathrm{d}s = \mathrm{o}(1).$$

On the other hand, (4.6) becomes

$$\int_0^1 (\langle\alpha(x_n)\dot{x}_n, \dot{\xi}_n\rangle + \langle\delta(x_n), \dot{\xi}_n\rangle\dot{t}_n + \langle\delta(x_n), \dot{x}_n\rangle\dot{\tau}_n - \beta(x_n)\dot{t}_n\dot{\tau}_n)\,\mathrm{d}s = \mathrm{o}(1).$$

Since $t_n = t - \tau_n$ and $x_n = x + \xi_n + \nu_n$, (4.4) and (4.5) imply that

$$\int_0^1 \langle\alpha(x_n)\dot{x}, \dot{\xi}_n\rangle\mathrm{d}s = \mathrm{o}(1), \qquad \int_0^1 (\langle\delta(x_n), \dot{\xi}_n\rangle\dot{t}_n + \langle\delta(x_n), \dot{x}_n\rangle\dot{\tau}_n)\,\mathrm{d}s = \mathrm{o}(1),$$

$$\int_0^1 \beta(x_n)\dot{t}\dot{\tau}_n\,\mathrm{d}s = \mathrm{o}(1).$$

Lastly, we obtain

$$\int_0^1 \langle\alpha(x_n)\dot{\xi}_n, \dot{\xi}_n\rangle\, \mathrm{d}s + \int_0^1 \beta(x_n)\dot{\tau}_n^2\, \mathrm{d}s = \mathrm{o}(1),$$

so (1.4) implies that $\xi_n \to 0$ strongly in $H^1(I, \mathbb{R}^N)$ and $\tau_n \to 0$ strongly in $H^1(I, \mathbb{R})$. □

Let us point out that in the (PS) proof we have used only the assumptions that $\alpha(x)$ and $\beta(x)$ are bounded from below and far from zero, while no control from above on the growth of the coefficients is required. On the other hand, this is not true any more in the proof of the (PS) condition for f^* (as well as in the proof of the geometrical estimates), so, in order to use the hypotheses (H_2), $(H_2)^*$ and (H_5), the following remarks will be useful.

Remark 4.3. Let us recall that for any $x_1, x_2 \in \mathcal{M}_0$, denoted by A_{x_1,x_2} the set of the piecewise smooth curves $\gamma : I \to \mathcal{M}_0$ such that $\gamma(0) = x_1$, $\gamma(1) = x_2$, it is

$$d(x_1, x_2) = \inf \left\{ \int_0^1 \sqrt{\langle\dot{\gamma}, \dot{\gamma}\rangle}\, \mathrm{d}s : \gamma \in A_{x_1,x_2} \right\}.$$

Then, by (H_3), taken $x_0 \in \mathcal{M}_0$, there exists $K > 0$ such that, if $x \in \Omega(P_0, P_1)$, there results

$$d(x(s), x_0) \le \int_0^1 \sqrt{\langle\dot{x}, \dot{x}\rangle}\, \mathrm{d}s + K \quad \text{for all } s \in I.$$

Hence, for any real number $p \ge 0$, it is

$$d^p(x(s), x_0) \le 2^p(\|\dot{x}\|^p + K^p) \quad \text{for all } s \in I,$$

where it is $\|\dot{x}\|^2 = \int_0^1 \langle\dot{x}, \dot{x}\rangle\, \mathrm{d}s$.

Remark 4.4. Let $a, b, q \ge 0$ be fixed. Then, by the Young inequality a positive constant $\gamma = \gamma(q)$ exists such that $a^q b \le a^{q+1} + \gamma b^{q+1}$.

Proposition 4.5. *Assume that $(H_2)^*$ holds. Then, the functional f^* satisfies the (PS) condition on Z^*.*

Proof. Let $(z_n)_{n\in\mathbb{N}}$ be a (PS) sequence in Z^*, i.e.,

$$\sup_{n\in\mathbb{N}} |f^*(z_n)| < +\infty, \tag{4.7}$$

$$\lim_{n\to+\infty} \mathrm{d}f^*(z_n) = 0. \tag{4.8}$$

As $\tau_n = t_n - T^* \in T_{t_n}W^*$, by (4.8) we deduce that there exists $\varepsilon_n \to 0$ as $n \to +\infty$ such that

$$\begin{aligned}\varepsilon_n\|\dot{t}_n - t^*\| &= \mathrm{d}f^*(z_n)[(0, \tau_n)] \\ &= 2\int_0^1 (\langle\delta(x_n), \dot{x}_n\rangle\dot{t}_n - \beta(x_n)\dot{t}_n^2)\, \mathrm{d}s - 2t^* \int_0^1 (\langle\delta(x_n), \dot{x}_n\rangle - \beta(x_n)\dot{t}_n)\, \mathrm{d}s.\end{aligned}$$

Then, by (2.5) the previous formula gives

$$\varepsilon_n \|\dot{t}_n - t^*\| = f^*(z_n) - \int_0^1 (\langle \alpha(x_n)\dot{x}_n, \dot{x}_n\rangle + \beta(x_n)\dot{t}_n^2)\, \mathrm{d}s \\ - 2t^* \int_0^1 (\langle \delta(x_n), \dot{x}_n\rangle - \beta(x_n)\dot{t}_n)\, \mathrm{d}s. \tag{4.9}$$

It is easy to see that by (1.5) and (1.6) and Remarks 4.3 and 4.4 there exist $R_1', R_2' > 0$ such that

$$\left|\int_0^1 \langle \delta(x_n), \dot{x}_n\rangle\, \mathrm{d}s\right| \le \int_0^1 (R_1 + R_2 d^q(x_n(s), x_0))|\dot{x}_n(s)|\, \mathrm{d}s \\ \le (R_1 + 2^q R_2(\|\dot{x}_n\|^q + K^q))\|\dot{x}_n\| \le R_1' + R_2'\|\dot{x}_n\|^{q+1} \tag{4.10}$$

and

$$\left|\int_0^1 \beta(x_n)\dot{t}_n\, \mathrm{d}s\right| \le (R_1 + 2^q R_2(\|\dot{x}_n\|^q + K^q))\|\dot{t}_n\| \le R_1' + R_2'(\|\dot{x}_n\|^{q+1} + \|\dot{t}_n\|^{q+1}).$$

Then, these last inequalities, (4.7) and (4.9) imply

$$\int_0^1 (\langle \alpha(x_n)\dot{x}_n, \dot{x}_n\rangle + \beta(x_n)\dot{t}_n^2)\, \mathrm{d}s \\ \le M^* - \varepsilon_n\|\dot{t}_n - t^*\| + 2|t^*|(2R_1' + 2R_2'\|\dot{x}_n\|^{q+1} + R_2'\|\dot{t}_n\|^{q+1})$$

for a suitable real constant M^*. As $q + 1 < 2$, (1.4) and the previous estimate assure that $(z_n)_{n\in\mathbb{N}}$ is bounded in Z^*; hence, up to a subsequence, $(z_n)_{n\in\mathbb{N}}$ converges weakly to a curve $z \in Z^*$. Arguing as in the second part of the proof of Proposition 4.1, we conclude that $(z_n)_{n\in\mathbb{N}}$ goes to z strongly in Z^*. □

Since the functionals f^0 and f^* are unbounded from above and from below on infinite-dimensional linear manifolds, the Rabinowitz saddle point theorem can not be directly applied, so we introduce a Galerkin approximation, more precisely a finite-dimensional approximation on the space of the time variable.

We consider the orthonormal basis $\{\sin(i\pi s)\}_{i\in\mathbb{N}}$ of H_0^1. For any $k \in \mathbb{N}$ we set

$$W_k^0 = H_k \oplus \mathbb{R}j_I, \qquad W_k^* = H_k + T^*,$$

where

$$H_k = \mathrm{span}\{\sin(i\pi s),\ \ i = 1, 2, \ldots, k\}.$$

Moreover, we set

$$Z_k^0 = \Omega(P_0, P_1) \times W_k^0, \qquad Z_k^* = \Omega(P_0, P_1) \times W_k^*$$

and

$$f_k^0 = f^0|_{Z_k^0}, \qquad f_k^* = f^*|_{Z_k^*}.$$

The following result allows to determinate the critical points of the strongly indefinite functional f^0 on Z^0 as limits of suitable sequences of critical points of the functionals f_k^0 on Z_k^0.

Proposition 4.6. *Assume that (1.4) holds. For any $k \in \mathbb{N}$ let $z_k \in Z_k^0$ be a critical point of f_k^0. Moreover, assume that two constants c_1 and c_2 exist, independent of k, such that*

$$c_1 \leq f_k^0(z_k) \leq c_2 \quad \textit{for all } k \in \mathbb{N}.$$

Then, up to subsequences, $(z_k)_{k\in\mathbb{N}}$ converges to a critical point z of f^0 such that $c_1 \leq f^0(z) \leq c_2$.

Proof. The same arguments of Proposition 4.2 prove that the sequence $(z_k)_{k\in\mathbb{N}}$ is bounded in Z^0; then, up to a subsequence, $z_k \rightharpoonup z$ weakly in Z^0. The remainder of the proof follows as in [5, Proposition 6.1]. □

An analogous result holds for the functional f^* on Z^*.

Proposition 4.7. *Assume that $(H_2)^*$ holds. For any $k \in \mathbb{N}$ let $z_k \in Z_k^*$ be a critical point of f_k^*. Moreover, assume that two constants $\bar{c}_1$ and $\bar{c}_2$ exist, independent of k, such that*

$$\bar{c}_1 \leq f_k^*(z_k) \leq \bar{c}_2 \quad \textit{for all } k \in \mathbb{N}.$$

Then, up to subsequences, $(z_k)_{k\in\mathbb{N}}$ converges to a critical point z of f^* such that

$$\bar{c}_1 \leq f^*(z) \leq \bar{c}_2.$$

Remark 4.8. It is possible to prove that the same results of Propositions 4.6 and 4.7 still hold if the critical levels $(f_k^0(z_k))_{k\in\mathbb{N}}$, respectively $(f_k^*(z_k))_{k\in\mathbb{N}}$, are bounded only from above.

Finally, we can prove the existence results stated in Theorems 1.3 and 1.4.

In the following, with a_i we denote suitable positive constants.

Proof of Theorem 1.3. Since our aim is to apply Theorem 3.2 to the functional f^0, let us point out that the same arguments used in the proof of Proposition 4.2 allows to state that f_k^0 satisfies the (PS) condition for all $k \in \mathbb{N}$. Now, taken $y \in \Omega(P_0, P_1) \cap C^1(I)$ and $R > 0$, let us define the following sets:

$$S^0 = \{(x, j_I) \in Z^0 : x \in \Omega(P_0, P_1)\} = \Omega(P_0, P_1) \times \{j_I\},$$
$$Q^0(R) = \{(y, t) \in Z^0 : \|t - j_I\|_0 \leq R\}$$

where $\|\cdot\|_0$ is defined in (2.7). Since $(\mathrm{d}/\mathrm{d}s) j_I(s) = 1$, by the hypothesis (H_2), Remark 4.3 and the estimate (4.10) there results

$$f^0(z) = \int_0^1 (\langle \alpha(x)\dot{x}, \dot{x}\rangle + 2\langle \delta(x), \dot{x}\rangle - \beta(x))\,\mathrm{d}s \geq \lambda\|\dot{x}\|^2 - a_1 - a_2\|\dot{x}\|^{q+1}$$

for all $z = (x, j_I) \in S^0$.

So, as $q+1<2$, there exists a constant $N>0$ such that

$$f^0(z) \geq \tfrac{1}{2}\lambda\|\dot{x}\|^2 - N \quad \text{for all } z \in S^0. \tag{4.11}$$

Therefore,

$$\inf f^0(S^0) \geq -N. \tag{4.12}$$

On the other hand, fixed $R>0$, for any $z=(y,t)\in Q^0(R)$ it is

$$f^0(z) = \int_0^1 (\langle\alpha(y)\dot{y},\dot{y}\rangle + 2\langle\delta(y),\dot{y}\rangle\dot{t} - \beta(y)\dot{t}^2)\,\mathrm{d}s \leq a_3 + a_4\|\dot{t}\| - \nu\|\dot{t}\|^2, \tag{4.13}$$

which gives

$$\sup f^0(Q^0(R)) < +\infty.$$

Straightforward calculations show that

$$|\|t-j_I\|_0 - 1| \leq \|t\|_0 \leq \|t-j_I\|_0 + 1 \quad \text{for all } t \in W^0. \tag{4.14}$$

So, $\|t-j_I\|_0 = R$ implies

$$|R-1| \leq \|t\|_0 \leq R+1 \quad \text{for all } z=(y,t)\in \partial Q^0(R).$$

Whence, since $\|\dot{t}\| = \|t\|_0$, by (4.13) it follows

$$f^0(z) \leq a_5 + a_6 R - \nu R^2 \quad \text{for all } z \in \partial Q^0(R). \tag{4.15}$$

By (4.12) and (4.15) we can choose $R^0>0$ so large that

$$\sup f^0(\partial Q^0(R^0)) < \inf f^0(S^0).$$

Then, by Theorem 3.2 for any $k\geq 1$ there exists a critical point z_k of f_k^0 such that

$$\inf f^0(S^0) \leq f_k^0(z_k) \leq \sup f^0(Q^0(R^0)).$$

Hence, Proposition 4.6 provides the existence of a critical point of the action functional f^0 on Z^0, i.e., a normal geodesic joining $\tilde{P}_0$ to $\tilde{P}_1$. □

In the sequel we will denote by $\bar{a}_i$ some positive constants independent of t^*.

Proof of Theorem 1.4. Taken $y \in \Omega(P_0,P_1)\cap C^1(I)$ and $R>0$, let us define the following sets:

$$S^* = \{(x,T^*)\in Z^* : x\in\Omega(P_0,P_1)\} = \Omega(P_0,P_1)\times\{T^*\},$$
$$Q^*(R) = \{(y,t)\in Z^* : \|t-T^*\|_0 \leq R\}.$$

Since $(\mathrm{d}/\mathrm{d}s)T^*(s) = t^*$, by $(H_2)^*$, Remarks 4.3 and 4.4 and arguing as in (4.10) we deduce that for all $z = (x, T^*) \in S^*$ it is

$$f^*(z) = \int_0^1 (\langle\alpha(x)\dot{x}, \dot{x}\rangle + 2\langle\delta(x), \dot{x}\rangle t^* - \beta(x)(t^*)^2)\,\mathrm{d}s$$
$$\geq \lambda\|\dot{x}\|^2 - \bar{a}_1|t^*|(\|\dot{x}\|^{q+1} + |t^*|\|\dot{x}\|^q) - \bar{a}_2(|t^*| + (t^*)^2)$$
$$\geq \lambda\|\dot{x}\|^2 - 2\bar{a}_1|t^*|\|\dot{x}\|^{q+1} - \bar{a}_3(|t^*|^{2+q} + (t^*)^2 + |t^*|) \tag{4.16}$$

with $q + 1 < 2$. Therefore, there exists a positive constant N^*, depending on t^*, such that

$$\inf f^*(S^*) \geq -N^*. \tag{4.17}$$

On the other hand, taken $R > 0$ it is

$$f^*(z) \leq \bar{a}_4 + \bar{a}_5\|\dot{t}\| - \nu\|\dot{t}\|^2 \quad \text{for all } z = (y, t) \in Q^*(R). \tag{4.18}$$

Hence,

$$\sup f^*(Q^*(R)) < +\infty.$$

Now, let us remark that if $t \in W^*$, then $t(0) = 0$ and $t(1) = t^*$, whence

$$\|t - T^*\|_0^2 = \|\dot{t}\|^2 - (t^*)^2 \tag{4.19}$$

and, in particular,

$$\|\dot{t}\|^2 = R^2 + (t^*)^2 \quad \text{if } z = (y, t) \in \partial Q^*(R), \text{ i.e., } \|t - T^*\|_0 = R. \tag{4.20}$$

Obviously, by (4.18) and (4.20) we have

$$\sup f^*(\partial Q^*(R)) \leq \bar{a}_4 + \bar{a}_5(R + |t^*|) - \nu(R^2 + (t^*)^2). \tag{4.21}$$

Hence, choosing a suitable R^* large enough, (4.17) and (4.21) imply that

$$\sup f^*(\partial Q^*(R^*)) < \inf f^*(S^*). \tag{4.22}$$

By Theorem 3.2 and Proposition 4.7 it follows that f^* has a critical point z such that

$$\inf f^*(S^*) \leq f^*(z) \leq \sup f^*(Q^*(R^*))$$

and, in particular, (4.16) implies

$$\inf f^*(S^*) \geq \inf_{x\in\Omega(P_0,P_1)} (\lambda\|\dot{x}\|^2 - 2\bar{a}_1|t^*|\|\dot{x}\|^{q+1} - 2\bar{a}_3(|t^*|^{2+q} + |t^*|)),$$

while (4.18) and $\|\dot{t}\| \geq |t^*|$ (see (4.19)) give

$$\sup f^*(Q^*(R^*)) \leq \bar{a}_4 + \bar{a}_5(R^* + |t^*|) - \nu(t^*)^2. \tag{4.23}$$

In order to study the causal character of the found geodesic, we need more information about the infimum of f^* on S^* and about a possible choice of the constant R^*.

Let us consider the map $\varphi^*(s) = \lambda s^2 - 2\bar{a}_1|t^*|s^{q+1} - 2\bar{a}_3(|t^*|^{2+q} + |t^*|)$ defined if $s \geq 0$. It is easy to see that φ^* attains its minimum in

$$s^* = \left(\frac{\bar{a}_1(q+1)|t^*|}{\lambda}\right)^{1/(1-q)}$$

and is strictly increasing in $[s^*, +\infty[$.

Let us remark that the assumptions (H_3) and (H_4) imply that $\bar{d} = d(P_0, P_1) > 0$ exists (independent of t^*) such that

$$\|\dot{x}\| \geq \bar{d} \quad \text{for all } \; x \in \Omega(P_0, P_1).$$

Then, if $|t^*|$ is small enough, it is $s^* < \bar{d}$ and $\varphi^*(\bar{d}) > 0$. Therefore,

$$\inf_{x \in \Omega(P_0,P_1)} (\lambda\|\dot{x}\|^2 - 2\bar{a}_1|t^*|\|\dot{x}\|^{q+1} - 2\bar{a}_3(|t^*|^{2+q} + |t^*|)) \geq \varphi^*(\bar{d}) > 0,$$

so, a previous estimate assures that the found geodesic is spacelike.

On the other hand, it can be proved that

$$\varphi^*(s^*) \geq -\bar{a}_6(|t^*|^{2/(1-q)} + |t^*|).$$

Whence, in (4.17) we can assume

$$N^* = \bar{a}_6(|t^*|^{2/(1-q)} + |t^*|) \tag{4.24}$$

and straightforward calculations show that in (4.22) we can fix the constant as $R^* = \bar{a}_7|t^*|^{1/(1-q)} + \bar{a}_8$. Hence, by (4.23) it is

$$f^*(z) \leq \bar{a}_9 + \bar{a}_{10}|t^*|^{1/(1-q)} - \nu(t^*)^2$$

and, if $1/(1-q) < 2$, i.e., $q < 1/2$, then $|t^*|$ large enough gives $f^*(z) < 0$ and the found geodesic is timelike. □

5. Multiplicity results

In this section we will prove the multiplicity results stated in Theorems 1.7 and 1.8. First, we will prove some technical lemmas.

Lemma 5.1. *There exists a continuous map* $\varrho^0 : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$z = (x, t) \in Z^0, \quad \|t - j_I\|_0 = \varrho^0(\|\dot{x}\|) \Rightarrow f^0(z) \leq -2N, \tag{5.1}$$

where N is the positive constant introduced in (4.12).

Proof. Let $z = (x, t) \in Z^0$; then by (1.4) and (1.5), (H_5) and Remark 4.3 it follows that

$$\begin{aligned} f^0(z) &\leq \int_0^1 (R_1 + R_2 d^p(x, x_0))\langle\dot{x}, \dot{x}\rangle \,\mathrm{d}s + 2\int_0^1 (R_1 + R_2 d^q(x, x_0))|\dot{x}||\dot{t}| \,\mathrm{d}s - \nu\|\dot{t}\|^2 \\ &\leq (R_1 + 2^p R_2(\|\dot{x}\|^p + K^p))\|\dot{x}\|^2 \\ &\quad + 2\int_0^1 (R_1 + 2^q R_2(\|\dot{x}\|^q + K^q))|\dot{x}||\dot{t}| \,\mathrm{d}s - \nu\|\dot{t}\|^2 \\ &\leq (R_1 + 2^p R_2(\|\dot{x}\|^p + K^p))\|\dot{x}\|^2 + 2(R_1 + 2^q R_2(\|\dot{x}\|^q + K^q))\|\dot{x}\|\|\dot{t}\| - \nu\|\dot{t}\|^2. \end{aligned}$$

Moreover, a particular case of the Young inequality implies that

$$f^0(z) \le (R_1 + 2^p R_2(\|\dot{x}\|^p + K^p))\|\dot{x}\|^2 + \frac{2}{\nu}(R_1 + 2^q R_2(\|\dot{x}\|^q + K^q))^2\|\dot{x}\|^2 - \frac{\nu}{2}\|\dot{t}\|^2, \tag{5.2}$$

and therefore, as $2q < p$ (see Remark 1.9) by (5.2) and (4.14) it follows that there exist b_1, $b_2 > 0$ such that

$$f^0(z) \le \|\dot{x}\|^2(b_1 + b_2\|\dot{x}\|^p) - \tfrac{1}{2}\nu(\|t - j_I\|_0 - 1)^2 \quad \text{for all } z = (x,t) \in Z^0. \tag{5.3}$$

Setting

$$\varrho^0(r) = 1 + \sqrt{\frac{2r^2(b_1 + b_2 r^p) + 4N}{\nu}}, \tag{5.4}$$

we have, clearly, that $\varrho^0 : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and

$$r^2(b_1 + b_2 r^p) - \tfrac{1}{2}\nu(\varrho^0(r) - 1)^2 = -2N \quad \text{for all } r \ge 0.$$

Hence, (5.1) follows by (5.3). □

Remark 5.2. By the definition (5.4) we have

$$\min_{r \in \mathbb{R}_+} \varrho^0(r) = \varrho^0(0) = 1 + \sqrt{\frac{4N}{\nu}} > 1.$$

In order to prove Theorem 1.7 we consider the following sets

$$S^0 = \Omega(P_0, P_1) \times \{j_I\}$$

and, fixed any integer $k \ge 1$,

$$C_k^0 = \{(x,t) \in Z_k^0 : \|t - j_I\|_0 = \varrho^0(\|\dot{x}\|)\}.$$

It is easy to show that (4.12) and Lemma 5.1 imply

$$\sup f_k^0(C_k^0) \le -2N < -N \le \inf f_k^0(S^0). \tag{5.5}$$

Lemma 5.3. *The set C_k^0 is a strong deformation retract of $Z_k^0 \backslash S^0$.*

Proof. The proof follows as in [5, Lemma 7.3]. □

Lemma 5.4. *Let $\mathcal{M}_0$ be 1-connected. For any $m \in \mathbb{N}$ there exists a compact subset K_m of Z_k^0 such that $\mathrm{cat}_{Z_k^0, C_k^0}(K_m) \ge m$.*

Proof. Consider the following sets:

$$B_k^0 = \{t \in W_k^0 : \|t - j_I\|_0 \le 1\}, \qquad \tilde{B}_k^0 = \Omega(P_0, P_1) \times B_k^0,$$
$$\Sigma_k^0 = \partial B_k^0 = \{t \in W_k^0 : \|t - j_I\|_0 = 1\}, \qquad \tilde{\Sigma}_k^0 = \Omega(P_0, P_1) \times \Sigma_k^0.$$

Let $m \in \mathbb{N}$. By (H_3) and (H_6), Theorem 3.10 implies the existence of a compact set $\Gamma_{k,m}$ in $\tilde{B}_k^0$ such that

$$\mathrm{cat}_{\tilde{B}_k^0,\tilde{\Sigma}_k^0}(\Gamma_{k,m}) \geq m. \tag{5.6}$$

Now, as ρ^0 is continuous and strictly positive (see Remark 5.2), arguing as in [5, Lemma 7.4] we can construct a retraction $\eta : Z_k^0 \to \tilde{B}_k^0$ and an homeomorphism $\Phi : Z_k^0 \to Z_k^0$ such that

$$\eta(x,t) = \begin{cases} (x,t) & \text{if}\,(x,t) \in \tilde{B}_k^0, \\ \left(x, \dfrac{t-j_I}{\|t-j_I\|_0} + j_I\right) & \text{if}\,(x,t) \notin \tilde{B}_k^0, \end{cases}$$

while

$$\Phi(x,t) = (x,\phi(x,t)) = (x,\varrho^0(\|\dot{x}\|)(t-j_I)+j_I).$$

It is easy to see that $\Phi(\tilde{\Sigma}_k^0) \subset C_k^0$ and $\eta \circ \Phi^{-1}(C_k^0) \subset \tilde{\Sigma}_k^0$. Hence, Proposition 3.7 and (5.6) provide that

$$\mathrm{cat}_{Z_k^0,C_k^0}(\Phi(\Gamma_{k,m})) \geq \mathrm{cat}_{\tilde{B}_k^0,\tilde{\Sigma}_k^0}(\Gamma_{k,m}) \geq m. \qquad \square$$

Remark 5.5. Let $\Gamma_{k,m}$ be as in the proof of Lemma 5.4. We can assume $\Gamma_{k,m} = V_m \times G_{k,m}$, where V_m is compact in $\Omega(P_0, P_1)$ while $G_{k,m}$ is compact in W_k^0. By Remark 3.11 the set V_m can be chosen independent of $k \geq 1$. By the definition of Φ it follows that also the set $K_m = \Phi(\Gamma_{k,m}) = V_m \times \phi(\Gamma_{k,m})$ has the spatial part independent of k.

Finally, we can state the multiplicity theorems given in Section 1. Even if their proofs are obtained as in [5], we outline them for completeness.

Proof of Theorem 1.7. Without loss of generality we can assume that $\mathcal{M}_0$ is a 1-connected manifold. In fact, if $\pi_1(\mathcal{M}_0)$ is finite and nontrivial, we can extend Lemma 5.4 by means of the universal covering of $\mathcal{M}_0$. On the contrary, if $\pi_1(\mathcal{M}_0)$ is infinite, it is enough to apply Theorem 1.3 at the functional restricted to each connected component. So, the existence of infinitely many solutions is stated; moreover, suitable a priori estimates prove that their energies diverge positively.

As the functional f_k^0 verifies the Palais–Smale condition on Z_k^0, (5.5) and Lemmas 5.3 and 5.4 allow to apply Theorem 3.8; then, a sequence of critical points $(z_k^m)_{m\geq 1}$ of f_k^0 exists such that

$$f_k^0(z_k^m) \geq \inf f_k^0(S^0), \qquad \lim_{m\to+\infty} f_k^0(z_k^m) = \sup f_k^0(Z_k^0) = +\infty.$$

Moreover, by Remark 3.9 there results

$$f_k^0(z_k^m) = \inf_{B\in F_{k,m}^0} \sup_{z\in B} f_k^0(z) \quad \text{for all}\, m \geq 1, \tag{5.7}$$

where

$$F_{k,m}^0 = \{B \subset Z_k^0 : B\,\text{closed}, \quad \mathrm{cat}_{Z_k^0,C_k^0}(B) \geq m\}.$$

Now, we claim that

(i) for all $m \geq 1$ there exists a constant $\gamma_m > 0$, independent of k, such that

$$f_k^0(z_k^m) \leq \gamma_m; \tag{5.8}$$

(ii) for all $c > 0$ there exists $m_c \in \mathbb{N}$, independent of k, such that

$$f_k^0(z_k^m) \geq \lambda c - N \quad \text{for all } m \geq m_c, \tag{5.9}$$

with λ as in (1.4) and N as in (4.12).

In fact, let $m \geq 1$ and $K_m = \Phi(\Gamma_{k,m})$ defined as in the proof of Lemma 5.4. By Remark 5.5, we can assume $K_m = V_m \times \phi(\Gamma_{k,m})$ with V_m compact subset of $\Omega(P_0, P_1)$. It is easy to see that, taken

$$\gamma_m = \max_{x \in V_m}(\|\dot{x}\|^2(b_1 + b_2\|\dot{x}\|^p)),$$

(5.3) implies

$$f_k^0(z) \leq \gamma_m \quad \text{for all } z = (x, t) \in K_m.$$

Then, the proof of (i) follows by (5.7).

Now, fix $c > 0$ and $k, m \geq 1$. Set

$$E^c = \{x \in \Omega(P_0, P_1) : \|\dot{x}\|^2 \leq 2c\}, \quad E_c = \{x \in \Omega(P_0, P_1) : \|\dot{x}\|^2 \geq 2c\}.$$

If $B \in F_{k,m}^0$ is such that

$$B \cap (E_c \times \{j_I\}) \neq \emptyset, \tag{5.10}$$

then by (4.11) it is

$$\sup f_k^0(B) \geq \lambda c - N. \tag{5.11}$$

On the other hand, let $B \in F_{k,m}^0$ be such that $B \cap (E_c \times \{j_I\}) = \emptyset$; then

$$B \subset (E^c \times \{j_I\}) \cup (Z_k^0 \backslash S^0).$$

By Propositions 3.4 and 3.6 and Lemma 5.3 it follows that

$$m \leq \text{cat}_{Z_k^0, C_k^0}(B) \leq \text{cat}_{\Omega(P_0,P_1)}(E^c). \tag{5.12}$$

As the assumptions (H_1) and (H_3) hold, the functional $\int_0^1 \langle \dot{x}, \dot{x} \rangle \, ds$ verifies the (PS) condition on $\Omega(P_0, P_1)$ (see [15]). Then, it is well known that

$$\text{cat}_{\Omega(P_0,P_1)}(E^c) < +\infty,$$

(for instance, cf. [18]). Hence, by (5.12), there exists $m_c \in \mathbb{N}$ such that for all $m \geq m_c$ and $B \in F_{k,m}^0$ (5.10) holds. Moreover, (5.9) follows by (5.11). So, by (5.8) and (5.9) there results

$$\lambda c - N \leq f_k^0(z_k^m) \leq \gamma_m \quad \text{for all } k \geq 1 \text{ if } m \geq m_c.$$

Whence, Proposition 4.6 implies that there exists a critical point z^m of f^0 such that

$$\lambda c - N \le f^0(z^m) \le \gamma_m. \tag{5.13}$$

Since $c > 0$ can be choosen arbitrarily large, thanks to (5.13) the previous arguments can be repeated and complete the proof. □

Proof of Theorem 1.8. Let $t^* \in \mathbb{R}$. Arguing as in the proof of Theorem 1.7, we can assume that $\mathcal{M}_0$ is 1-connected.

Fixed $k \ge 1$, it is easy to define a continuous map $\varrho^* : \mathbb{R}_+ \to \mathbb{R}_+$ such that it results that

$$\sup f_k^*(C_k^*) \le -2N^* < -N^* \le \inf f_k^*(S^*) \quad \text{if } t^* \ne 0,$$

while

$$\sup f_k^*(C_k^*) \le -1 < 0 \le \inf f_k^*(S^*) \quad \text{if } t^* = 0$$

with

$$S^* = \Omega(P_0, P_1) \times \{T^*\}, \qquad C_k^* = \{(x,t) \in Z_k^* : \|t - T^*\|_0 = \varrho^*(\|\dot{x}\|)\}.$$

Moreover, C_k^* is a strong deformation retract of $Z_k^* \backslash S^*$ and for all $m \ge 1$ there exists a compact subset K_m^* of Z_k^* such that

$$\mathrm{cat}_{Z_k^*, C_k^*}(K_m^*) \ge m, \tag{5.14}$$

where K_m^* has the spatial part V_m^* independent of k and t^* (these results can be proved as in Lemmas 5.3 and 5.4 by replacing j_I with T^* and ϱ^0 with ϱ^*).

Assume $t^* \ne 0$ (the proof is simpler if $t^* = 0$) and fix $m \ge 1$ and K_m^* such that (5.14) holds. Let us point out that K_m^* is defined as in the proof of Lemma 5.4. More precisely, taken

$$\tilde{B}_k^* = \Omega(P_0, P_1) \times \{t \in W_k^* : \|t - T^*\|_0 \le 1\},$$
$$\Phi^* : (x,t) \in Z_k^* \mapsto (x, \varrho^*(\|\dot{x}\|)(t - T^*) + T^*) \in Z_k^*,$$

it is $K_m^* = \Phi^*(\Gamma_{k,m}^*)$, where $\Gamma_{k,m}^*$ is a suitable compact subset of $\tilde{B}_k^*$.

Hence, if $(x,t) \in K_m^*$ it is easy to see that

$$\|t - T^*\|_0 \le \varrho^*(\|\dot{x}\|).$$

Then, by (4.19), there results

$$|t^*| \le \|\dot{t}\| \le \varrho^*(\|\dot{x}\|) + |t^*| \quad \text{for all } z = (x,t) \in K_m^*. \tag{5.15}$$

Obviously, (1.4) and the Hölder inequality give

$$f_k^*(z) \le \int_0^1 \langle \alpha(x)\dot{x}, \dot{x}\rangle \,\mathrm{d}s + 2\left(\int_0^1 \langle \delta(x), \dot{x}\rangle^2 \,\mathrm{d}s\right)^{1/2} \|\dot{t}\| - \nu\|\dot{t}\|^2.$$

Therefore, since V_m^* is bounded in $\Omega(P_0, P_1)$, there exist two positive constants p_m and q_m, independent of k and t^*, such that (5.15) implies

$$\sup f_k^*(K_m^*) \leq p_m + q_m|t^*| - \nu(t^*)^2. \tag{5.16}$$

On the other hand, reasoning as in the proof of (5.9), for any $c > 0$ there exists $m_c \in \mathbb{N}$, independent of k, such that, defined

$$F_{k,m}^* = \{B \subset Z_k^* : B \text{ closed}, \operatorname{cat}_{Z_k^*, C_k^*}(B) \geq m\},$$

for all $B \in F_{k,m}^*$ it is $B \cap (E_c \times \{T^*\}) \neq \emptyset$ and then

$$\inf_{B \in F_{k,m}^*} \sup_{z \in B} f_k^*(z) \geq \varphi^*(\sqrt{2c}) \quad \text{for all } m \geq m_c, \tag{5.17}$$

where φ^* is defined in the proof of Theorem 1.3. By (5.16) and (5.17) the same arguments used in the proof of Theorem 1.7 show the existence of a monotonically increasing sequence $(c_m^*)_{m\geq 1}$ of critical levels of f^* such that

$$\varphi^*(\sqrt{2c}) \leq c_m^* \leq p_m + q_m|t^*| - \nu(t^*)^2. \tag{5.18}$$

Clearly, (5.18) implies the existence of infinitely many spacelike geodesics joining $\tilde{P}_0$ to $\tilde{P}_1^*$.

Lastly, we prove that (1.8) holds. First of all we remark that if two integers $m, h \geq 1$ exist such that $c_m^* = c_{m+1}^* = \cdots = c_{m+h}^*$, arguing as in [11], there exist at least h distinct critical points at level c_m^* (e.g., cf. [4, Lemma 5.9]). Moreover, for all $m \geq 1$ there exists $T_m > 0$ such that for all $|t^*| \geq T_m$ there results

$$p_m + q_m|t^*| - \nu(t^*)^2 < 0$$

and therefore by (5.18) f^* has at least m distinct critical points having negative energy. □

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